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BRIEF COMMUNICATION

THERMOCAPILLARY MIGRATION OF BUBBLES AT LARGE REYNOLDS NUMBERS

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Abstract—The thermocapillary movement of bubbles has been investigated for large Reynolds numbers. A numerical analysis of the flow field around a bubble has been carried out for arbitrary Marangoni (Ma) numbers. For small Ma the flow field has been calculated using a matched asymptotic procedure. A comparison is made of the nondimensional bubble velocity and the temperature gradient at the bubble surface, calculated numerically, with analytical results for large and small values of Ma, and with numerical results obtained previously. The expansion for large Ma has also been proposed by Balasubramaniam and Subramanian (1996). For Ma$\gg$1 the presence of both the thermal boundary layer and the wake impose limitations in the numerical calculations. The calculations have been carried out for values of Ma significantly larger than those previously obtained. © 1998 Elsevier Science Ltd. All rights reserved

Key Words: bubbles, Marangoni convection, asymptotic analysis, numerical analysis, boundary layers, wakes, thermocapillarity

1. INTRODUCTION

The migration of bubbles in a liquid environment, produced by thermocapillary forces in the presence of temperature gradients, is a problem of interest in processing of materials in zero gravity. Much work has been done on the subject that has been reviewed by Wozniak *et al.* (1988) and Subramanian (1992). Much of the work has been dedicated to both low Reynolds, Re, and Marangoni, Ma, numbers; among them, the pioneering work of Young *et al.* (1959) for the vanishing convection should be mentioned. However, as pointed out by Subramanian (1992), significant convective effects may appear in many practical situations. Crespo and Manuel (1983) and independently Balasubramaniam and Chai (1987) showed that for Ma$\ll$1, the solution of Young *et al.* is an exact solution of the momentum equation for any Re. Crespo and Jiménez-Fernández (1991) carried out an analysis for the limit of both Re$\gg$1 and Ma$\gg$1, and found thermal and viscous boundary layers surrounding the bubble, and obtained an expression for the bubble velocity that differed from that of Young *et al.* for negligible convection by a numerical factor; similar results were obtained by Balasubramaniam and Subramanian (1996) using a slightly different approach. In a previous paper, Crespo and Jiménez-Fernández (1992) showed that in the limit of Ma$\gg$1, there is a thermal boundary layer that can be solved separately yielding a boundary condition for the momentum equation that has to be solved numerically for arbitrary Re; in the limit of Re$\gg$1 the solution mentioned above is recovered, and for Re$\ll$1 an expansion of the velocity field in terms of Gegenbauer functions and Legendre polynomials was performed, giving again a bubble velocity that differed from that of Young *et al.* for negligible convection, by a numerical factor. Balasubramaniam and Subramanian (1996) showed that there was an error in the numerical procedure of Crespo and Jiménez-Fernández (1992), and that this factor was consequently not correct. Szymczyk and Siekmann (1988) performed a numerical analysis retaining convective effects; however, the number of results they obtain under the conditions that will be examined here is rather limited.

In this work the limit of Re$\gg$1 and arbitrary Ma will be examined. It is shown that the viscous boundary layer introduces only a small perturbation on the velocity field, that essentially corresponds to the irrotational flow around a sphere. Only the corresponding linear equation for temperature has to be solved. The bubble velocity is obtained from an integral condition, introduced by Crespo and Jiménez-Fernández (1992), that has to be satisfied in order to avoid a singularity of the viscous boundary layer velocity at the rear of the bubble. As indicated previously, the solution is already known analytically in both the limits Ma = 0 and infinity, besides, Balasubramaniam and Subramanian (1996), also gave the asymptotic behavior for Ma$\gg$1. However, the asymptotic behavior for Ma$\ll$1 is not known, and here an analysis is presented examining this limit, that is similar to that performed by Subramanian (1981) for the limit in which both Ma, Re$\ll$1. It is the purpose of this paper to examine the transition from small to large values of Ma and establish the range of validity of the analytical solutions. Balasubramaniam (1995) obtained numerical results for some values of Ma, however, he concentrated more on the numerical aspects and did not compare with the asymptotic behaviors in the limiting situations, although he obtained the correct limiting solutions for Ma = 0 and infinity. In order to establish this comparison it has been necessary to obtain more numerical results, particularly in the region of large Ma; in this work, values of Ma as large as 50,000 are reached (that can be extended to 500,000 if a non-uniform grid is used), whereas Balasubramaniam (1995) only reached values of 2000.

In this work, as in most of the previously mentioned references, it is assumed that surface tension effects are large enough so that the bubble remains spherical; small deformations are calculated by Balasubramaniam and Chai (1987), when both Weber (Wb) and Capillary (Ca) numbers are small, however, this assumption is more difficult to satisfy in our case, as Re$\gg$1 and Wb = Ca $\times$ Re. It is also assumed that the bubble does not expand or contract, because the external temperature variations are small enough to induce small changes in the gas density and, besides, the heat transfer to the bubble is very small, as a result of the small heat conductivity of the gas, that in this analysis it is assumed to be negligible compared to that of the liquid.

2. GOVERNING EQUATIONS

The nondimensional conservation equations for the liquid in a reference frame fixed to a bubble are:

conservation of mass,

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

conservation of momentum,

$$\mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \frac{1}{\mathrm{Re}} \Delta \mathbf{v}, \tag{2}$$

conservation of energy,

$$1 + \mathbf{v} \cdot \nabla T = \frac{1}{\mathrm{Pe}} \Delta T. \tag{3}$$

Where $\mathbf{v}$, p and T are the non-dimensional velocity, manometric pressure and temperature respectively. Distances have been non-dimensionalized with the bubble radius, R, the velocity with the bubble velocity, V_∞, the temperature with $1/R(\mathrm{d}\,T_\infty/\mathrm{d}x)$, $\mathrm{d}T_\infty\,/\,\mathrm{d}x$ is the gradient of the temperature at infinity, and the manometric pressure with $\rho\, V_\infty{}^2$, where ρ is the density of the liquid. The Reynolds and Peclet numbers are respectively $\mathrm{Re} = \rho V_\infty R/\mu$ and $\mathrm{Pe} = V_\infty R/\kappa$, where μ is the viscosity and κ the thermal diffusivity of the liquid. The boundary conditions are:

$$v_\theta = \sin\theta, \qquad v_r = -\cos\theta, \qquad p = 0, \qquad T = r\cos\theta, \qquad \text{for } r = \infty, \tag{4}$$

$$v_r = 0 \qquad \frac{\partial T}{\partial r} = 0, \qquad \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = \frac{1}{v_\infty}\frac{\partial T}{\partial \theta}, \qquad \text{for } r = 1, \tag{5}$$

where axial symmetry is assumed, spherical polar coordinates have been used, and the parameter v_∞ is $(\mu V_\infty)/(\sigma_T R\, \mathrm{d}T_\infty/\mathrm{d}x)$, where σ_T is minus the derivative of surface tension with temperature, that is supposed to be positive. v_∞ represents the non-dimensional bubble velocity, that has to be calculated. The Marangoni number is then given by Ma = Pe/v_∞.

For large values of Re, as shown by Crespo and Jiménez-Fernández (1991), the velocity field is given in a first approximation by the irrotational solution:

$$\mathbf{v}_0 = \nabla\phi, \qquad \text{where } \phi = -\left(r + \frac{1}{2r^2}\right)\cos\theta. \tag{6}$$

This solution in general does not satisfy (only in the limit Pe = Ma = 0) the last boundary condition in [5], and a hydrodynamic boundary layer is required. However, this boundary condition is on the velocity gradient, not on the velocity itself, and as the boundary layer thickness is small, only a small perturbation on the velocity is expected. The appropriate boundary layer variables are defined by:

$$r = 1 + \frac{y}{\mathrm{Re}^{1/2}}, \qquad v_\theta = v_{\theta_0} + \frac{u}{\mathrm{Re}^{1/2}}, \qquad v_r = \frac{v}{\mathrm{Re}^{1/2}}. \tag{7}$$

Introducing these variables in [2], using [6] for v_{θ_0}, and assuming the pressure to be that of the irrotational flow, the following equation is obtained for the azimuthal velocity perturbation (see also Levich (1962)):

$$\frac{3}{2}\sin\theta\frac{\partial u}{\partial\theta} + \frac{3}{2}u\cos\theta - 3y\cos\theta\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \tag{8}$$

At $y = \infty$, $u = 0$, and the boundary condition [5] is now:

$$\frac{\partial u}{\partial y} = 3\sin\theta + \frac{1}{v_\infty}\frac{\partial T}{\partial\theta}, \qquad \text{at } y = 0, \tag{9}$$

Besides, it should also be imposed that, for $\theta = 0$, $u = 0$. Crespo and Jiménez-Fernández (1991) introduced a function, $U(\theta)$, representing the total flux of the perturbed velocity, whose value can be obtained from integration of equation [8] across the boundary layer:

$$U = \int_0^\infty u\mathrm{d}y = -\frac{2}{3\,(\sin\theta)^3}\int_0^\theta \left(\frac{\partial u}{\partial y}\right)_{y=0} (\sin\alpha)^2 \mathrm{d}\alpha. \tag{10}$$

This solution gives a singularity for $\theta = \pi$, unless the integral vanishes. Imposing this condition and using the boundary condition [9], the following equation is obtained to calculate the non-dimensional value of the bubble velocity:

$$v_\infty = -\frac{1}{4}\int_0^\pi \left(\frac{\partial T}{\partial\theta}\right)_{y=0} \sin^2\theta\, \mathrm{d}\theta. \tag{11}$$

This result was also obtained by Balasubramaniam and Subramanian (1996) using viscous dissipation arguments. In the limiting situation of Pe = Ma = 0 the solution of the temperature field is $T = -\phi$, where ϕ is given in [6], that was substituted in [11] yields the classical result $v_\infty = \frac{1}{2}$ of Young *et al.* (1959), that can also be obtained from [9] with $u = 0$. In the opposite limit of Ma$\gg$1 there is a thermal boundary layer, and Crespo and Jiménez-Fernández (1991) and Balasubramaniam and Subramanian (1996) show that $v_\infty = (8–3\log 3)/24 = 0.196$.

Here, the solution for Re = ∞ and arbitrary Pe will be obtained. Assuming that the velocity field is given in first approximation by [6], the temperature field will be obtained from [3] with $\mathbf{v} = \mathbf{v}_0$, and boundary conditions on temperature are given in [4] and [5]: $\partial T/\partial r = 0$ at $r = 1$, $T = r\cos\theta$ at $r = \infty$. When the temperature field is known, the value of v_∞ is obtained from [11].

3. SOLUTION FOR SMALL PECLET NUMBERS

The solution for small Ma or Pe is obtained using a matched asymptotic expansion procedure, applied in a classical way. The procedure is quite similar to that used by Subramanian

(1981), although in this case it is easier, because only the temperature has to be calculated, as the velocity field is known. Here, only the final results are given and the details of the calculation can be found in that reference. The inner variables are r and θ, and the outer variables are $r_1 = r\,\mathrm{Pe}$ and θ. The inner temperature field is given by:

$$T = t_0 + t_1\,\mathrm{Pe} + t_2\,\mathrm{Pe}^2 + O(\mathrm{Pe}^3), \tag{12}$$

where,

$$t_0 = \left(r + \frac{1}{2r^2}\right)\cos\theta, \tag{13}$$

$$t_1 = \frac{1}{6r} - \frac{1}{24r^4} + \left(\frac{2}{9r^3} - \frac{1}{3r} - \frac{1}{12r}\right)P_2(\cos\theta), \tag{14}$$

$$\begin{aligned} t_2 = &-\frac{1}{6} + \left(-\frac{13}{60} + \frac{11}{360r^2} - \frac{1}{120r^3} - \frac{1}{45r^5} + \frac{1}{80r^6}\right)\cos\theta \\ &+ \left(\frac{1}{20} - \frac{1}{15r^2} + \frac{1}{20r^3} + \frac{1}{40r^4} - \frac{1}{30r^5} + \frac{1}{120r^6}\right)P_3(\cos\theta), \end{aligned} \tag{15}$$

and P_i is the Legendre polynomial of order i. The terms in the order of 0 and 1 are the same ones calculated by Subramanian (1981), except there is a factor 2 difference in t_1 because we are expanding in terms of the Peclet number instead of the Marangoni number. The outer temperature field is given by:

$$T = \frac{H_0}{\mathrm{Pe}} + H_1 + H_2\,\mathrm{Pe} + H_3\,\mathrm{Pe}^2 + O(\mathrm{Pe}^3), \tag{16}$$

where

$$H_0 = r_1\cos(\theta), \qquad H_1 = H_2 = 0, \tag{17}$$

and,

$$H_3 = -\frac{\cos(\theta)}{2r_1^2} + \exp\left[-\left(\frac{r_1}{2}\right)(1 + \cos(\theta))\right]\left(\frac{2 + r_1}{2r_1^2}\cos(\theta) + \frac{1}{3r_1}\right). \tag{18}$$

The migration velocity is obtained from [11] and [12],

$$v_\infty = \frac{1}{2} - \frac{49}{720}\mathrm{Pe}^2 + \cdots = \frac{1}{2} - \frac{49}{2880}\mathrm{Ma}^2 + \cdots, \tag{19}$$

where the relationship $\mathrm{Ma} = \mathrm{Pe}/v_\infty$ has been used. The correction to the classical solution, $v_\infty = \frac{1}{2}$, is $O(\mathrm{Pe}^2)$. Subramanian (1981), in the limit of small Re, gives for the correction term ($-301/14{,}400\ \mathrm{Ma}^2$), that is larger than the one calculated here. It also appeared in Subramanian (1981), that a regular expansion procedure will give the correct results for t_0 and t_1, however, the constant term in t_2 will not be obtained, and the regular expansion would fail because the condition at infinity is not satisfied. Then, the outer solution, given by [16]–[18], is necessary to match the inner solution as r goes to infinity. However, the regular expansion without the constant term in [15] is able to give the correct terminal velocity, so that the outer temperature field is not strictly necessary in this case if one is only interested in the result given by [19]; this is probably due to the fact that at infinity $T = O(r)$ is much larger than the constant term. The residual obtained from substituting [12] into [3] contains terms of $O(\mathrm{Pe}^3/r)$, so that it is likely that, if it is attempted to obtain a higher order term in the inner expansion, it will probably diverge at infinity like $O(r)$, and the outer expansion will be needed.

4. NUMERICAL SOLUTION AND COMPARISON WITH OTHER RESULTS

The discretization of [3] has been made using centred differences for all the derivatives. The azimuthal and radial coordinates have been divided in $M_\theta = 90$ and $M_r = 300$ elements, respectively. A sensitivity analysis has also been made using a smaller number of elements, and the results have not changed significantly, except for large Pe, as explained below. The matrix of the coefficients, of order $M_\theta M_r$, is sparse and banded, and has been converted to upper triangular; therefore, the values of temperature at each node have been calculated using a direct method.

Infinity is located at $r = 10$, for Pe of the order of one or smaller. For large values of Pe, there is a thin thermal boundary layer near $r = 1$, of thickness $O(1/\mathrm{Pe}^{1/2})$, so that the radial elements have to be very small. For this case the outer boundary has been located at a distance:

$$R_{\mathrm{ext}} = 1 + N/\mathrm{Pe}^{1/2}, \qquad [20]$$

where N has been given different values: 10, 20 30; no significant variation of the results was observed as N is changed. If $R_{\mathrm{ext}} > 10$, the outer boundary is located at $r = 10$. In general, R_{ext} is not large enough for condition [4] for the temperature to be satisfied, and instead, the following condition, obtained from the outer solution of [3] without the diffusion term, given by Balasubramaniam and Subramanian (1996), has been used:

$$T = R_{\mathrm{ext}}\cos(\theta) + \int_{R_{\mathrm{ext}}}^{\infty} \frac{1}{\tilde{r}^3 - 1}\left(\frac{3\Psi}{\tilde{r}^2 - \frac{1}{\tilde{r}}} - 1\right)\frac{1}{\left(1 - 2\frac{\Psi}{\tilde{r}^2 - \frac{1}{\tilde{r}}}\right)^{1/2}}\mathrm{d}\tilde{r}, \qquad [21]$$

where Ψ is the stream function corresponding to the velocity field of [6]: $\Psi = \frac{1}{2}\sin^2(\theta)(r^2\text{-}1/r)$.

Another problem for large Pe comes up at the rear of the bubble, where there are large values of the thermal gradient. This large value of $\partial T/\partial\theta$ can be explained considering [3] without the diffusive term; the equilibrium between the convective and unsteady terms gives $\partial T/\partial\theta = -1/v_\theta$ over the bubble surface, so that the temperature gradient becomes singular both at the front and rear stagnation points; the singularity at the front is cancelled by the thermal boundary layer of thickness $1/\mathrm{Pe}^{1/2}$, however, at the back, the boundary layer thickness becomes infinite, originating a thermal wake, where the diffusive term in the r direction is very small to cancel the effect of the unsteady term. Balasubramaniam and Subramanian (1996) have treated analytically this singularity by considering within the thermal wake a very small inner region around $\theta = \pi$ of thickness $1/\mathrm{Pe}^{1/2}$, where diffusive effects in the θ direction are important.

For values of Pe > 10,000 (Ma > 50,000) oscillations of the temperature field of wavelength of the order of the grid size appear near the wake, that for larger Pe extended to the whole flow field; however, in spite of these oscillations, the values of v_∞ seem to be correctly predicted, even if Ma is of the order of 10^6. A similar problem has been found by Balasubraniam (1995). Using a non-uniform grid the calculations can be extended up to Ma of the order of 500,000, without significant oscillations of the temperature, however, in this case the truncation error is an order of magnitude larger than with the uniform grid.

In figure 1 are given the values of v_∞ as a function of Ma and compared with the asymptotic expansion for v_∞ obtained by Balasubramaniam and Subramanian (1996) for large values of Ma,

$$v_\infty = \left(\frac{1}{3} - \frac{\log 3}{8}\right) - 0.1369\frac{1}{\mathrm{Ma}^{1/2}}\log\left(\frac{1}{\mathrm{Ma}^{1/2}}\right) + 0.6578\frac{1}{\mathrm{Ma}^{1/2}} \qquad [22]$$

and with expression [19], valid for small Ma. Presented also are the numerical results of Balasubraniam (1995). The values of v_∞ calculated are very similar to those calculated here, although slightly smaller for large Ma. Equation [19] is in acceptable agreement with the numerical calculations for Ma < 2, where the maximum relative error is of the order of 10%. Equation [22] is in acceptable agreement with the numerical results for values of Ma > 15 where the relative error is also of the order of 10%. From the results shown in figure 1 it looks as though with further approximations in [19] and [22] their ranges of agreement with the numeri-

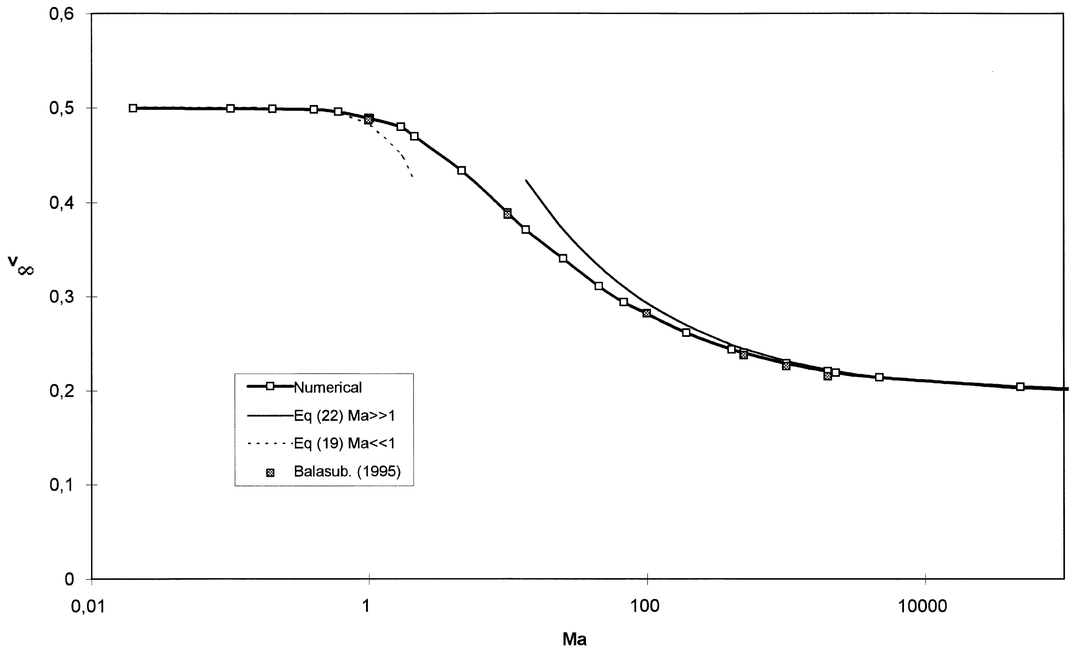

Figure 1. Non-dimensional bubble velocity as function of Marangoni number Ma. Comparison with numerical calculations of Balasubramaniam (1995), and asymptotic expansions for large and small Ma.

cal results could be significantly extended. As an example: by adding the term –0.42/Ma to the r.h.s. of [22], and the term 0.0007 Ma^4 to the r.h.s of [19], the difference between the numerical results and the results of [22] and [19] would be less than 7% in their respective ranges: $\infty > Ma > 4$, $4 > Ma > 0$; although it should be stressed that the correct calculation of the next order terms is a very laborious task.

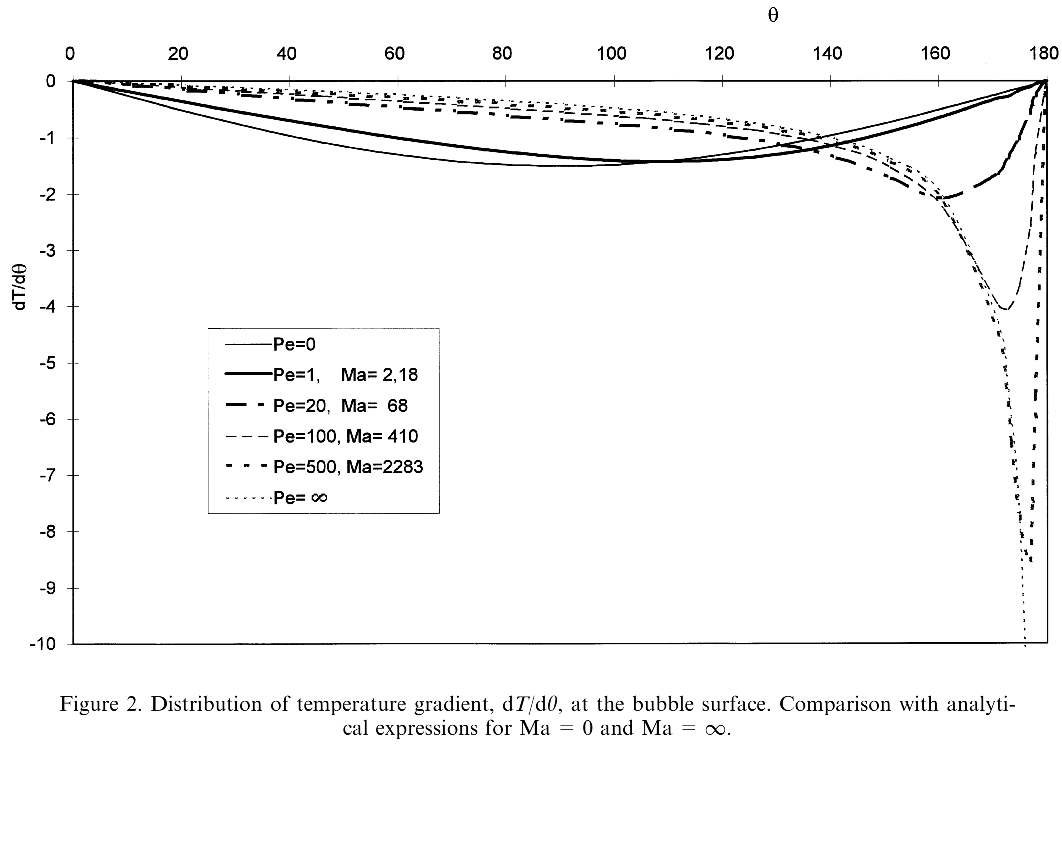

Figure 2. Distribution of temperature gradient, $dT/d\theta$, at the bubble surface. Comparison with analytical expressions for Ma = 0 and Ma = ∞.

From the results of Szymczyk and Siekmann (1988) only one value could be obtained for finite Pe (Pe = 10, Ma = 31) and large Re (Re = 100), giving $v_\infty = 0.35$, that is in acceptable agreement with the results shown in figure 1, that gives $v_\infty = 0.32$.

In figure 2 are given the distributions of the temperature gradient at the surface of the bubble for different values of Pe. Also are drawn the corresponding distributions for Pe = 0 and ∞, that can be obtained from Crespo and Jiménez-Fernández (1990) and Balasubramaniam and Subramanian (1996), and are given by:

$$\frac{\mathrm{d}T}{\mathrm{d}\theta} = -\frac{3}{2}\sin(\theta), \qquad r = 1, \qquad \mathrm{Pe} = 0, \tag{23}$$

$$\frac{\mathrm{d}T}{\mathrm{d}\theta} = -\frac{1}{6}\frac{[5 + 3\cos(\theta)]\sin(\theta)}{[1 + \cos(\theta)][2 + \cos(\theta)]}, \qquad r = 1, \qquad \mathrm{Pe} = \infty. \tag{24}$$

For Pe of order 1 the distribution is similar to that of Pe = 0, but the minimum is slightly larger than $-\frac{3}{2}$ and displaced to values of θ larger than 90°. For larger Pe the minimum is displaced further back to larger values of θ, however it gets smaller than $-\frac{3}{2}$ and decreases as Ma increases. For Pe of order 20, the distribution becomes very similar to that of Pe = ∞, up to the position of the minimum; for larger Pe, the distributions follow the same trend, giving smaller minima, that are getting closer to $\theta = 180°$, where $\mathrm{d}\,T/\mathrm{d}\theta = 0$, showing clearly how the gradient in the back becomes singular for Pe = ∞. From the analysis of Balasubramaniam and Subramanian (1996) it is easy to obtain that the thickness of the region between the minimum and $\theta = \pi$ is $\pi - \theta_{\min} = 1.29/\mathrm{Pe}^{0.5} = 2.92/\mathrm{Ma}^{0.5}$, that is approximately satisfied by the results of figure 2, even for Pe as low as 20.

5. CONCLUDING REMARKS

An asymptotic analysis for small Ma and a numerical analysis for arbitrary Ma have been carried out of the flow field around a bubble, whose movement is induced by thermocapillary forces, in the limit of high Reynolds numbers. The nondimensional bubble velocity and the temperature gradient at the bubble surface, calculated numerically, are presented and compared with analytical results for both large and small Ma, and with numerical results obtained independently by Balasubramaniam (1995). The expansion for large Ma has been proposed by Balasubramaniam and Subramanian (1996). Difficulties in the numerical integration appear for Ma$\gg$1 because the presence of both the thermal boundary layer and the wake. The boundary layer effect is solved by imposing the boundary condition closer to the bubble, and using the outer non-diffusive solution. The wake imposes a limitation on the maximum value of Ma. The numerical results show how the large gradients in the back are formed.

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REFERENCES

Balasubramaniam, R. and Chai, A. T. (1987) Thermocapillary migration of droplets: an exact solution for small marangoni numbers. *J. Colloid Interface Sci.* **119,** 531–538.

Balasubramaniam, R. and Subramanian, R. S. (1996) Thermocapillary bubble migration-thermal boundary layers for large Marangoni numbers. *Int. J. Multiphase Flow* **22,** 593–612.

Balasubramaniam, R. (1995) Thermocapillary bubble migration-solution of the energy equation for potential- flow approximated velocity fields. *Computational Fluid Dynamics* **3,** 407–414.

Crespo, A. and Jiménez-Fernández, J. 1991 Thermocapillary migration of bubbles at moderately large Reynolds numbers. *Microgravity Science and Thecnology* **4**, 79–80. (The whole paper is published by Springer Verlag as proceedings of the IUTAM Symposium, Bremen, September 1991.)

Crespo, A. and Jiménez-Fernández, J. (1992) Thermocapillary migration of bubbles: a semi-analytic solution for large Marangoni numbers. *Proceedings of the 8th European Symposium on Materials and Fluid Sciences in Microgravity*, ESA SP-333. Brussels, Belgium. pp. 193–196.

Crespo, A. and Manuel, F. (1983) Bubble motion under reduced gravity. *Proceedings of the 4th European Symposium on Materials Sciences Under Microgravity*. Madrid, Spain. 45–49.

Levich, V. G. (1962) *Physicochemical Hydrodynamics*. Prentice-Hall, Englewood Cliffs, NJ.

Szymczyk, J. and Siekmann, J. (1988) Numerical calculation of the thermocapillary motion of a bubble under microgravity. *Chem. Eng. Comm.* **69,** 129–147.

Subramanian, R. S. (1981) Slow migration of a bubble in a thermal gradient. *AIChE Journal* **27,** 646–654.

Subramanian, R. S. (1992) The motion of bubbles and drops in reduced gravity. In *Transport Processes in Bubbles, Drops and Particles*, ed. R. P. Chhabra and D. Dekee, pp. 1–42. Hemisphere, New York.

Wozniak, G., Siekmann, J. and Srulijes, J. (1988) Thermocapillary bubble and drop dynamics under reduced gravity—survey and prospects. *Z. Flugwiss. Weltraumforsch* **12,** 137–144.

Young, N. O., Goldstein, J. S. and Block, M. J. (1959) The motion of bubbles in a vertical temperature gradient. *Journal of Fluid Mechanics* **6,** 350–366.